

# A SEARS-TYPE SELF-ADJOINTNESS RESULT FOR DISCRETE MAGNETIC SCHRÖDINGER OPERATORS

**ABSTRACT.** In the context of a weighted graph with vertex set  $V$  and bounded vertex degree, we give a sufficient condition for the essential self-adjointness of the operator  $\Delta_\sigma + W$ , where  $\Delta_\sigma$  is the magnetic Laplacian and  $W: V \rightarrow \mathbb{R}$  is a function satisfying  $W(x) \geq -q(x)$  for all  $x \in V$ , with  $q: V \rightarrow [1, \infty)$ . The condition is expressed in terms of completeness of a metric that depends on  $q$  and the weights of the graph. The main result is a discrete analogue of the results of I. Oleinik and M. A. Shubin in the setting of non-compact Riemannian manifolds.

## 1. INTRODUCTION AND THE MAIN RESULT

**1.1. The setting.** Let  $G = (V, E)$  be an infinite graph without loops and multiple edges between vertices. By  $V = V(G)$  and  $E = E(G)$  we denote the set of vertices and the set of unoriented edges of  $G$  respectively. In what follows, the notation  $m(x)$  indicates the degree of a vertex  $x$ , that is, the number of edges that meet at  $x$ . We assume that  $G$  has bounded vertex degree: there exists a constant  $N > 0$  such that

$$m(x) \leq N, \quad \text{for all } x \in V. \quad (1.1)$$

In what follows,  $x \sim y$  indicates that there is an edge that connects  $x$  and  $y$ . We will also need a set of oriented edges

$$E_0 := \{[x, y], [y, x] : x, y \in V \text{ and } x \sim y\}. \quad (1.2)$$

The notation  $e = [x, y]$  indicates an oriented edge  $e$  with starting vertex  $o(e) = x$  and terminal vertex  $t(e) = y$ . The definition (1.2) means that every unoriented edge in  $E$  is represented by two oriented edges in  $E_0$ . Thus, there is a two-to-one map  $p: E_0 \rightarrow E$ . For  $e = [x, y] \in E_0$ , we denote the corresponding reverse edge by  $\hat{e} = [y, x]$ . This gives rise to an involution  $e \mapsto \hat{e}$  on  $E_0$ .

To help us write formulas in unambiguous way, we fix an orientation on each edge by specifying a subset  $E_s$  of  $E_0$  such that  $E_0 = E_s \cup \widehat{E_s}$  (disjoint union), where  $\widehat{E_s}$  denotes the image of  $E_s$  under the involution  $e \mapsto \hat{e}$ . Thus, we may identify  $E_s$  with  $E$  by the map  $p$ .

In the sequel, we assume that  $G$  is connected, that is, for any  $x, y \in V$  there exists a path  $\gamma$  joining  $x$  and  $y$ . Here,  $\gamma$  is a sequence  $x_1, x_2, \dots, x_n \in V$  such that  $x = x_1$ ,  $y = x_n$ , and  $x_j \sim x_{j+1}$  for all  $1 \leq j \leq n-1$ .

In what follows,  $C(V)$  is the set of complex-valued functions on  $V$ , and  $C(E_s)$  is the set of functions  $Y: E_0 \rightarrow \mathbb{C}$  such that  $Y(e) = -Y(\hat{e})$ . The notations  $C_c(V)$  and  $C_c(E_s)$  denote the sets of finitely supported elements of  $C(V)$  and  $C(E_s)$  respectively.

In the sequel, we assume that  $V$  is equipped with a weight  $w: V \rightarrow \mathbb{R}^+$ . By  $\ell_w^2(V)$  we denote the space of functions  $f \in C(V)$  such that  $\|f\| < \infty$ , where  $\|f\|$  is the norm corresponding to the inner product

$$(f, g) := \sum_{x \in V} w(x) f(x) \overline{g(x)}. \quad (1.3)$$

Additionally, we assume that  $E$  is equipped with a weight  $a: E_0 \rightarrow \mathbb{R}^+$  such that  $a(e) = a(\widehat{e})$  for all  $e \in E_0$ . This makes  $G = (G, w, a)$  a weighted graph with weights  $w$  and  $a$ .

**1.2. Magnetic Schrödinger operator.** Let  $U(1) := \{z \in \mathbb{C}: |z| = 1\}$  and  $\sigma: E_0 \rightarrow U(1)$  with  $\sigma(\widehat{e}) = \overline{\sigma(e)}$  for all  $e \in E_0$ , where  $\bar{z}$  denotes the complex conjugate of  $z \in \mathbb{C}$ .

We define the magnetic Laplacian  $\Delta_\sigma: C(V) \rightarrow C(V)$  on the graph  $(G, w, a)$  by the formula

$$(\Delta_\sigma u)(x) = \frac{1}{w(x)} \sum_{e \in \mathcal{O}_x} a(e)(u(x) - \sigma(\widehat{e})u(t(e))), \quad (1.4)$$

where  $x \in V$  and

$$\mathcal{O}_x := \{e \in E_0: o(e) = x\}. \quad (1.5)$$

For the case  $a \equiv 1$  and  $w \equiv 1$ , the definition (1.4) is the same as in [9]. For the case  $\sigma \equiv 1$ , see [30] and [32].

Let  $W: V \rightarrow \mathbb{R}$ , and consider a Schrödinger-type expression

$$Hu := \Delta_\sigma u + Wu. \quad (1.6)$$

Let  $q: V \rightarrow [1, \infty)$ , and assume that  $W$  satisfies

$$W(x) \geq -q(x), \quad \text{for all } x \in V. \quad (1.7)$$

In the sequel, we will need the notion of weighted distance on  $G$ . Let  $w$  and  $a$  be as in (1.4) and let  $q$  be as in (1.7). We define the weighted distance  $d_{w,a;q}$  on  $G$  as follows:

$$d_{w,a;q}(x, y) := \inf_{\gamma \in \Gamma_{x,y}} L_{w,a;q}(\gamma), \quad (1.8)$$

where  $\Gamma_{x,y}$  is the set of all paths  $\gamma: x = x_1, x_2, \dots, x_n = y$  such that  $x_j \sim x_{j+1}$  for all  $1 \leq j \leq n-1$ , and the length  $L_{w,a;q}(\gamma)$  is computed as follows:

$$L_{w,a;q}(\gamma) = \sum_{j=1}^{n-1} \frac{\min\{w^{1/2}(x_j), w^{1/2}(x_{j+1})\} \cdot \min\{q^{-1/2}(x_j), q^{-1/2}(x_{j+1})\}}{\sqrt{a([x_j, x_{j+1}])}}.$$

In the case  $q \equiv 1$ , the weighted distance (1.8) was defined in [4].

We say that the metric space  $(G, d_{w,a;q})$  is complete if every Cauchy sequence of vertices has a limit in  $V$ .

**1.3. Statement of the main result.** We now state the main result.

**Theorem 1.4.** *Assume that  $(G, w, a)$  is an infinite, connected, oriented, and weighted graph. Assume that  $G$  has bounded vertex degree. Assume that  $W$  satisfies (1.7) and  $q: V \rightarrow [1, \infty)$  satisfies*

$$|q^{-1/2}(t(e)) - q^{-1/2}(o(e))| \leq C \left( \frac{\min\{w(t(e)), w(o(e))\}}{a(e)} \right)^{1/2}, \quad (1.9)$$

for all  $e \in E_s$ , where  $C$  is a constant.

Additionally, assume that  $(G, d_{w,a;q})$  is a complete metric space. Then, the operator  $H|_{C_c(V)}$  is essentially self-adjoint in  $\ell_w^2(V)$ .

*Remark 1.5.* The origin of the result presented in Theorem 1.4 can be traced back to the paper [25] by D. B. Sears concerning the essential self-adjointness of  $(-\Delta + W)|_{C_c^\infty(\mathbb{R}^n)}$  in  $L^2(\mathbb{R}^n)$ . Here,  $\Delta$  is the standard Laplacian on  $\mathbb{R}^n$  and  $-q \leq W \in L_{\text{loc}}^\infty(\mathbb{R}^n)$ , where  $q$  is a radially symmetric function on  $\mathbb{R}^n$  satisfying properties analogous to those of Theorem 1 in the present paper (with “completeness” replaced by the divergence of  $\int_0^\infty q^{-1/2}(r) dr$ , where  $r = r(x)$  is the Euclidean distance between  $x \in \mathbb{R}^n$  and  $0 \in \mathbb{R}^n$ ). We should mention that the paper [25] followed an idea of E. C. Titchmarsh [31]. More recently, I. Oleinik [23, 24] gave a sufficient condition for the essential self-adjointness of  $(\Delta_M + W)|_{C_c^\infty(M)}$  in  $L^2(M)$ , where  $\Delta_M$  is the scalar Laplacian on a Riemannian manifold  $M$  and  $-q \leq W \in L_{\text{loc}}^\infty(M)$ . Here,  $q$  is a function on  $M$  satisfying properties analogous to those of Theorem 1 in the present paper. Oleinik’s proof was simplified by M. A. Shubin [26], and the result was extended to magnetic Schrödinger operators in [27]. Theorem 1.4 of the present paper is a discrete analogue of the mentioned results of Oleinik and Shubin.

*Remark 1.6.* Assuming (1.1), the completeness of  $(G, d_{w,a;1})$ , and

$$(Hu, u) \geq k\|u\|^2, \quad \text{for all } u \in C_c(V), \quad (1.10)$$

where  $k$  is a constant independent of  $u$ , the essential self-adjointness of  $H|_{C_c(V)}$  was established in [21, Theorem 1.3]. If  $q(x) \equiv c_0$ , where  $c_0$  is a constant, then the operator  $H|_{C_c(V)}$ , with  $W$  as in (1.7), satisfies (1.10). However, there are operators  $H$  that satisfy the hypotheses of Theorem 1.4 but do not satisfy (1.10), as illustrated by the example below.

*Example .* Consider  $G = (V, E)$  with  $V = \{1, 2, 3, \dots\}$  and  $E = \{[n, n+1]: n \in V\}$ . Define  $a([n, n+1]) = 1$  and  $w(n) = 1$ , for all  $n \in V$ . Let  $H$  be as in (1.6) with  $\sigma([n, n+1]) = 1$  and  $W(n) = -n^2$ , for all  $n \in V$ . It is easy to see that for every  $k \in \mathbb{R}$ , there exists a function  $u \in C_c(V)$  such that the inequality (1.10) is not satisfied. Thus, the operator  $H$  is not semi-bounded from below, and we cannot use [21, Theorem 1.3]. Turning to hypotheses of Theorem 1.4, note that  $W$  satisfies (1.7) with  $q(n) = n^2$ . It is easy to see that  $q^{-1/2} = n^{-1}$  satisfies (1.9) with  $C = 1$ . Fix  $K_1 \in V$ , and let  $K > K_1$ . For  $x_1 = K_1$  and  $x = K$ , by (1.8) we have

$$d_{w,a;q}(x_1, x) = \sum_{n=K_1}^{K-1} \frac{1}{n+1} \rightarrow \infty, \quad \text{as } K \rightarrow \infty.$$

Thus, the metric  $d_{w,a;q}$  is complete, and by Theorem 1.4 the operator  $H|_{C_c(V)}$  is essentially self-adjoint in  $\ell_w^2(V)$ .

*Remark 1.7.* Thanks to assumption (1.10), the proof of [21, Theorem 1.3] reduced to showing that if  $u \in \text{Dom}(H_{\max})$ , with  $H_{\max}$  as in Section 2 below, and  $(H + \lambda)u = 0$  with sufficiently large  $\lambda > 0$ , then  $u = 0$ . To this end, a sequence of cut-off functions was constructed and a “summation by parts” method was used. In the absence of assumption (1.10), the essential self-adjointness can be established by showing that  $H_{\max}$  is symmetric. This requires an approach different from [21]: in the present paper, we consider the sum  $J_s$  that incorporates the metric  $d_{w,a;q}$  (see (3.20) below) and show that  $J_s \rightarrow 0$  as  $s \rightarrow +\infty$ . A key ingredient in this endeavor, not present in [21], is the estimate (3.2) for  $d_\sigma u$ , where  $u \in \text{Dom}(H_{\max})$ . The estimate (3.2) is a discrete analogue of [27, Lemma 4.3].

*Remark 1.8.* For studies of the operator (1.4) with  $a \equiv 1$ ,  $\sigma \equiv 1$ , and  $w \equiv m$ , see, for instance, [3] and [22]. For general information concerning magnetic Laplacian on graphs, see [20] and [29]. For a proof the discrete analogue of Kato’s inequality, see [9].

For the problem of self-adjoint realization of the operator (1.6) and its special cases ( $a \equiv 1$ ,  $\sigma \equiv 1$ ,  $w \equiv 1$ , and  $W \equiv 0$ ), see, for instance, [4], [5], [11], [12], [15], [17], [18], [32], [33], and [35]. We should mention that the authors of [12] and [17, 18] worked in the setting of discrete sets, a more general context than locally finite graphs. For a study of the essential self-adjointness of discrete Laplace operator on forms, see [19].

The problem of stochastic completeness of graphs is considered in [7], [33], [35], and [36]. In the setting of Dirichlet forms on discrete sets, stochastic completeness is studied in [12], [17], and [18]. For another approach to stochastic completeness on discrete sets, see [13]. For a study of random walks on infinite graphs, see [6], [8], [34], and references therein.

For studies of essential self-adjointness of Schrödinger operators in the context of non-compact Riemannian manifolds, see, for instance, [1], [2], [10], [23], [24], [26], [27], and [28].

## 2. PRELIMINARIES

In what follows, the deformed differential  $d_\sigma: C(V) \rightarrow C(E_s)$  is defined as

$$(d_\sigma u)(e) := \overline{\sigma(e)}u(t(e)) - u(o(e)), \quad \text{for all } u \in C(V), \quad (2.1)$$

where  $\sigma$  is as in (1.4).

The deformed co-differential  $\delta_\sigma: C(E_s) \rightarrow C(V)$  is defined as

$$(\delta_\sigma Y)(x) := \frac{1}{w(x)} \sum_{\substack{e \in E_s \\ t(e)=x}} \sigma(e)a(e)Y(e) - \frac{1}{w(x)} \sum_{\substack{e \in E_s \\ o(e)=x}} a(e)Y(e), \quad (2.2)$$

for all  $Y \in C(E_s)$ , where  $\sigma$ ,  $w$ , and  $a$  are as in (1.4).

In the case  $\sigma \equiv 1$ , the definitions (2.1) and (2.2) give us the standard differential  $d$  and standard co-differential  $\delta$ , respectively.

Let  $\sigma$  be as in (1.4). For a function  $u \in C(V)$ , we define  $u_\sigma^\sharp \in C(E_s)$  by the formula

$$u_\sigma^\sharp(e) := \frac{\sigma(e)u(t(e)) + u(o(e))}{2}, \quad \text{for all } e \in E_s. \quad (2.3)$$

For  $\sigma \equiv 1$  in (2.3), we define  $u^\sharp(e) := u_1^\sharp(e)$ .

In what follows, for  $x \in V$ , we define

$$\mathcal{S}_x := \{e \in E_s : o(e) = x \text{ or } t(e) = x\}. \quad (2.4)$$

The proofs of the following two lemmas are straightforward computations based on the definitions of  $d$ ,  $d_\sigma$ ,  $\delta$  and  $\delta_\sigma$ . For detailed proofs in the case  $\sigma \equiv 1$  see [19, Lemma 3.1].

**Lemma 2.1.** *For all  $u \in C(V)$  and all  $v \in C(V)$ , the following equality holds:*

$$d_{\overline{\sigma}}(uv) = (d_{\overline{\sigma}}u)v^\sharp + u_\sigma^\sharp dv, \quad (2.5)$$

where  $d_{\overline{\sigma}}$  is as in (2.1) with  $\sigma(e)$  replaced by  $\overline{\sigma(e)}$ ,  $u_\sigma^\sharp$  is as in (2.3), and  $v^\sharp$  is as in (2.3) with  $\sigma \equiv 1$ .

**Lemma 2.2.** *For all  $u \in C(V)$  and all  $Y \in C(E_s)$ , the following equality holds:*

$$(\delta(u_\sigma^\sharp Y))(x) = u(x)(\delta_\sigma Y)(x) - \frac{1}{2w(x)} \sum_{e \in \mathcal{S}_x} a(e)Y(e)(d_{\overline{\sigma}}u)(e), \quad (2.6)$$

where  $d_{\overline{\sigma}}$  is as in (2.1) with  $\sigma(e)$  replaced by  $\overline{\sigma(e)}$ ,  $u_\sigma^\sharp$  is as in (2.3), and  $\mathcal{S}_x$  is as in (2.4).

**Lemma 2.3.** *Assume that  $\phi \in C(V)$  is real-valued. Then*

$$(\phi^\sharp(e))^2 \leq (\phi^2)^\sharp(e), \quad \text{for all } e \in E_s. \quad (2.7)$$

**Proof** By (2.3) with  $\sigma \equiv 1$ , for all  $e \in E_s$  we have

$$(\phi^2)^\sharp(e) - (\phi^\sharp(e))^2 = \left( \frac{\phi(t(e)) - \phi(o(e))}{2} \right)^2 \geq 0,$$

which gives (2.7).  $\square$

Let  $\ell_a^2(E_s)$  denote the space of functions  $F \in C(E_s)$  such that  $\|F\| < \infty$ , where  $\|F\|$  is the norm corresponding to the inner product

$$(F, G) := \sum_{e \in E_s} a(e)F(e)\overline{G(e)}.$$

It is easy to check the following equality:

$$(d_\sigma u, Y) = (u, \delta_\sigma Y), \quad \text{for all } u \in \ell_w^2(V), Y \in C_c(E_s), \quad (2.8)$$

where  $(\cdot, \cdot)$  on the left-hand side (right-hand side) denotes the inner product in  $\ell_a^2(E_s)$  (in  $\ell_w^2(V)$ ).

A computation shows that the following equality holds:

$$\delta_\sigma d_\sigma u = \Delta_\sigma u, \quad \text{for all } u \in C(V). \quad (2.9)$$

For the proofs of (2.8) and (2.9), see, for instance, [21, Section 3]. The following lemma follows easily from (2.9) and (2.8).

**Lemma 2.4.** *The operator  $\Delta_\sigma|_{C_c(V)}$  is symmetric in  $\ell_w^2(V)$ :*

$$(\Delta_\sigma u, v) = (u, \Delta_\sigma v), \quad \text{for all } u, v \in C_c(V).$$

We now give the definitions of minimal and maximal operators associated with the expression (1.6). We define the operator  $H_{\min}$  by the formula

$$H_{\min}u := Hu, \quad \text{Dom}(H_{\min}) := C_c(V). \quad (2.10)$$

Since  $W$  is real-valued, the following lemma follows easily from Lemma 2.4.

**Lemma 2.5.** *The operator  $H_{\min}$  is symmetric in  $\ell_w^2(V)$ .*

We define  $H_{\max} := (H_{\min})^*$ , where  $T^*$  denotes the adjoint of operator  $T$ . We also define  $\mathcal{D} := \{u \in \ell_w^2(V) : Hu \in \ell_w^2(V)\}$ .

For a proof of the following lemma, see, for instance, [21, Lemma 3.7].

**Lemma 2.6.** *The following hold:  $\text{Dom}(H_{\max}) = \mathcal{D}$  and  $H_{\max}u = Hu$  for all  $u \in \mathcal{D}$ .*

### 3. PROOF OF THEOREM 1.4

In this section, we will adapt the technique of Shubin [27].

Let  $H_{\min}$  and  $H_{\max}$  be as in Section 2. By Lemma 2.5 we know that  $H_{\min}$  is symmetric. Thus, by Kato [16, Problem V.3.10],  $H_{\min}$  is essentially self-adjoint if and only if

$$(H_{\max}u, v) = (u, H_{\max}v), \quad \text{for all } u, v \in \text{Dom}(H_{\max}). \quad (3.1)$$

The following proposition provides useful information about  $\text{Dom}(H_{\max})$ .

**Proposition 3.1.** *If  $u \in \text{Dom}(H_{\max})$ , then*

$$\begin{aligned} & \sum_{e \in E_s} \min\{q^{-1}(o(e)), q^{-1}(t(e))\} a(e) |(d_\sigma u)(e)|^2 \\ & \leq 2((2C^2N + 1)\|u\|^2 + \|Hu\|\|u\|), \end{aligned} \quad (3.2)$$

where  $H$  is as in (1.6),  $N$  is as in (1.1), and  $C$  is as in (1.9).

In the proof of Proposition 3.1, we will use a sequence of cut-off functions. Fix a vertex  $x_0 \in V$ , and define

$$\chi_n(x) := \left( \left( \frac{2n - d_{w,a;1}(x_0, x)}{n} \right) \vee 0 \right) \wedge 1, \quad x \in V, \quad n \in \mathbb{Z}_+, \quad (3.3)$$

where  $d_{w,a;1}(x_0, x)$  is as in (1.8) with  $q \equiv 1$ .

In the case  $w \equiv 1$  and  $a \equiv 1$ , the sequence (3.3) was constructed in [19, Proposition 3.2]. Denote

$$B_n^{w,a}(x_0) := \{x \in V : d_{w,a;1}(x_0, x) \leq n\}. \quad (3.4)$$

The sequence  $\{\chi_n\}_{n \in \mathbb{Z}_+}$  satisfies the following properties: (i)  $0 \leq \chi_n(x) \leq 1$ , for all  $x \in V$ ; (ii)  $\chi_n(x) = 1$  for  $x \in B_n^{w,a}(x_0)$  and  $\chi_n(x) = 0$  for  $x \notin B_{2n}^{w,a}(x_0)$ ; (iii) for all  $x \in V$ , we have

$\lim_{n \rightarrow \infty} \chi_n(x) = 1$ ; (iv) the functions  $\chi_n$  have finite support; and (v) the functions  $d\chi_n$  satisfy the inequality

$$|(d\chi_n)(e)| \leq \frac{d_{w,a;1}(o(e), t(e))}{n}.$$

It is easy to see that the properties (i)–(iii) and (v) hold. By hypothesis, we know that  $(G, d_{w,a;q})$  is a complete metric space and, thus, balls with respect to  $d_{w,a;q}$  are finite; see, for instance, [21, Section 6.1]. Let  $B_{2n}^{w,a;q}(x_0)$  be as in (3.4) with  $d_{w,a;1}$  replaced by  $d_{w,a;q}$ . Since  $q \geq 1$  it follows that  $B_{2n}^{w,a}(x_0) \subseteq B_{2n}^{w,a;q}(x_0)$ . Thus, property (iv) is a consequence of property (ii) and the finiteness of  $B_{2n}^{w,a}(x_0)$ .

### Proof of Proposition 3.1

Let  $u \in \text{Dom}(H_{\max})$  and let  $\phi \in C_c(V)$  be a real-valued function. Define

$$I := \left( \sum_{e \in E_s} a(e) |(d_\sigma u)(e)|^2 (\phi^2)^\sharp(e) \right)^{1/2}, \quad (3.5)$$

where  $f^\sharp(e)$  is as in (2.3) with  $\sigma \equiv 1$ .

We will first show that

$$I^2 \leq |(\phi^2 H u, u)| + (\phi^2 q u, u) + 2I \left( \sum_{e \in E_s} a(e) |(d\phi)(e)|^2 |(\bar{u})_\sigma^\sharp(e)|^2 \right)^{1/2}, \quad (3.6)$$

where  $f_\sigma^\sharp(e)$  is as in (2.3), and  $\bar{z}$  is the conjugate of  $z \in \mathbb{C}$ .

Using (2.6), the equality  $\Delta_\sigma u = H u - W u$ , and

$$(d_{\bar{\sigma}}(\phi^2 \bar{u}))(e) = \overline{(d_\sigma u)(e)} (\phi^2)^\sharp(e) + 2(\bar{u})_\sigma^\sharp(e) \phi^\sharp(e) (d\phi)(e),$$

we have

$$\begin{aligned} \delta \left( (\phi^2 \bar{u})_\sigma^\sharp d_\sigma u \right) (x) &= \phi^2(x) \overline{u(x)} (H u - W u)(x) \\ &\quad - \frac{1}{2w(x)} \sum_{e \in \mathcal{S}_x} a(e) |(d_\sigma u)(e)|^2 (\phi^2)^\sharp(e) \\ &\quad - \frac{1}{w(x)} \sum_{e \in \mathcal{S}_x} a(e) (d_\sigma u)(e) (\bar{u})_\sigma^\sharp(e) \phi^\sharp(e) (d\phi)(e). \end{aligned} \quad (3.7)$$

Since  $\phi$  has finite support, using the definition of  $\delta$  it follows that

$$\sum_{x \in V} \left( w(x) \delta \left( (\phi^2 \bar{u})_\sigma^\sharp d_\sigma u \right) (x) \right) = 0. \quad (3.8)$$

Multiplying both sides of (3.7) by  $w(x)$ , summing over  $x \in V$ , and using (3.8), we get

$$\begin{aligned} \frac{1}{2} \sum_{x \in V} \sum_{e \in \mathcal{S}_x} a(e) |(d_\sigma u)(e)|^2 (\phi^2)^\sharp(e) &= (\phi^2 H u, u) - (\phi^2 W u, u) \\ &\quad - \sum_{x \in V} \sum_{e \in \mathcal{S}_x} a(e) (d_\sigma u)(e) (\bar{u})_\sigma^\sharp(e) \phi^\sharp(e) (d\phi)(e). \end{aligned} \quad (3.9)$$

Rewriting the double sum on the left-hand side of (3.9) as the sum over  $E_s$ , taking real parts on both sides of (3.9), and using (1.7), we have

$$\begin{aligned}
& \sum_{e \in E_s} a(e) |(d_\sigma u)(e)|^2 (\phi^2)^\sharp(e) = \operatorname{Re} (\phi^2 H u, u) - (\phi^2 W u, u) \\
& - \operatorname{Re} \sum_{x \in V} \sum_{e \in \mathcal{S}_x} a(e) (d_\sigma u)(e) (\bar{u})_\sigma^\sharp(e) \phi^\sharp(e) (d\phi)(e) \\
& \leq |(\phi^2 H u, u)| + (\phi^2 q u, u) \\
& + 2 \sum_{e \in E_s} a(e) |(d_\sigma u)(e)| |(\bar{u})_\sigma^\sharp(e)| |\phi^\sharp(e)| |(d\phi)(e)|,
\end{aligned}$$

which, after applying Cauchy–Schwarz inequality and (2.7), gives (3.6).

Let  $\chi_n$  be as in (3.3) and let  $q$  be as in (1.7). Define

$$\phi_n(x) := \chi_n(x) q^{-1/2}(x). \quad (3.10)$$

By property (iv) of  $\chi_n$  it follows that  $\phi_n$  has finite support. By property (i) of  $\chi_n$  and since  $q \geq 1$ , we have

$$0 \leq \phi_n(x) \leq q^{-1/2}(x) \leq 1, \quad \text{for all } x \in V. \quad (3.11)$$

By property (iii) of  $\chi_n$  we have

$$\lim_{n \rightarrow \infty} \phi_n(x) = q^{-1/2}(x), \quad \text{for all } x \in V. \quad (3.12)$$

By (2.5), (1.9), properties (i) and (v) of  $\chi_n$ , the inequality  $q \geq 1$ , and (1.8), we have

$$\begin{aligned}
|(d\phi_n)(e)| &= |(d\chi_n)(e) (q^{-1/2})^\sharp(e) + (\chi_n)^\sharp(e) (dq^{-1/2})(e)| \\
&\leq \left( \frac{1}{n} + C \right) \frac{\min\{w^{1/2}(o(e)), w^{1/2}(t(e))\}}{\sqrt{a(e)}},
\end{aligned} \quad (3.13)$$

where  $C$  is as in (1.9).

We also have

$$|(\bar{u})_\sigma^\sharp(e)|^2 \leq \frac{|u(o(e))|^2 + |u(t(e))|^2}{2}. \quad (3.14)$$

By (3.13), (3.14), and (1.1) we get

$$\begin{aligned}
& \left( \sum_{e \in E_s} a(e) |(d\phi_n)(e)|^2 |(\bar{u})_\sigma^\sharp(e)|^2 \right)^{1/2} \\
& \leq \frac{1}{\sqrt{2}} \left( \frac{1}{n} + C \right) \left( \sum_{e \in E_s} |u(o(e))|^2 w(o(e)) + \sum_{e \in E_s} |u(t(e))|^2 w(t(e)) \right)^{1/2} \\
& \leq \frac{1}{\sqrt{2}} \left( \frac{1}{n} + C \right) (2N \|u\|^2)^{1/2} = \left( \frac{1}{n} + C \right) \sqrt{N} \|u\|.
\end{aligned} \quad (3.15)$$



By (3.6) with  $\phi = \phi_n$ , (3.15), and (3.11), we obtain

$$I_n^2 \leq \|Hu\|\|u\| + \|u\|^2 + 2I_n \left( \frac{1}{n} + C \right) \sqrt{N}\|u\|, \quad (3.16)$$

for all  $u \in \text{Dom}(H_{\max})$ , where  $I_n$  is as in (3.5) with  $\phi = \phi_n$ .

Using the inequality  $ab \leq \frac{a^2}{4} + b^2$  in the third term on the right-hand side of (3.16) and rearranging, we obtain

$$I_n^2 \leq 2 \left( \|Hu\|\|u\| + \left( 2N \left( \frac{1}{n} + C \right)^2 + 1 \right) \|u\|^2 \right). \quad (3.17)$$

Letting  $n \rightarrow \infty$  in (3.17) and using (3.12) together with Fatou's lemma, we get

$$\sum_{e \in E_s} a(e) |(d_\sigma u)(e)|^2 (q^{-1})^\sharp(e) \leq 2 \left( \|Hu\|\|u\| + (2NC^2 + 1) \|u\|^2 \right). \quad (3.18)$$

Since

$$\min\{q^{-1}(o(e)), q^{-1}(t(e))\} \leq (q^{-1})^\sharp(e), \quad \text{for all } e \in E_s,$$

the inequality (3.2) follows directly from (3.18).  $\square$

In the sequel, we will prove (3.1). Let  $d_{w,a;q}$  be as in (1.8). Fix  $x_0 \in V$  and define

$$P(x) := d_{w,a;q}(x_0, x), \quad x \in V. \quad (3.19)$$

In what follows, for a function  $f: V \rightarrow \mathbb{R}$  we define  $f^+(x) := \max\{f(x), 0\}$ .

Let  $u, v \in \text{Dom}(H_{\max})$  and let  $s > 0$ . Define

$$J_s := \sum_{x \in V} \left( 1 - \frac{P(x)}{s} \right)^+ \left( (Hu)(x) \overline{v(x)} - u(x) \overline{(Hv)(x)} \right) w(x), \quad (3.20)$$

where  $P$  is as in (3.19),  $H$  is as in (1.6), and  $\bar{z}$  denotes the conjugate of  $z \in \mathbb{C}$ .

Since  $(G, d_{w,a;q})$  is a complete metric space, by [21, Section 6.1] it follows that the set

$$U_s := \{x \in V : P(x) \leq s\}$$

is finite. Thus, for all  $s > 0$ , the summation in (3.20) is performed over finitely many vertices.

**Lemma 3.2.** *Let  $J_s$  be as in (3.20). Then*

$$\lim_{s \rightarrow +\infty} J_s = (Hu, v) - (u, Hv). \quad (3.21)$$

**Proof** For all  $x \in V$ , as  $s \rightarrow +\infty$ , the summand in (3.20) converges to

$$((Hu)(x) \overline{v(x)} - u(x) \overline{(Hv)(x)}) w(x).$$

Additionally, for all  $x \in V$  and  $s > 0$ , the summand in (3.20) is estimated from above by

$$|(Hu)(x)| |\overline{v(x)}| w(x) + |u(x)| |\overline{(Hv)(x)}| w(x).$$

Since  $u, v \in \text{Dom}(H_{\max})$ , by Lemma 2.6 we have  $Hu \in \ell_w^2(V)$  and  $Hv \in \ell_w^2(V)$ . Hence, by Cauchy–Schwarz inequality it follows that

$$\sum_{x \in V} |(Hu)(x)| |\overline{v(x)}| w(x) < +\infty \quad \text{and} \quad \sum_{x \in V} |u(x)| |\overline{(Hv)(x)}| w(x) < +\infty.$$

Thus, by dominated convergence theorem we obtain (3.21).  $\square$

**Lemma 3.3.** *Let  $J_s$  be as in (3.20) and let  $N$  be as in (1.1). Then*

$$\begin{aligned} |J_s| &\leq \frac{\sqrt{N}}{s} \|v\| \left( \sum_{e \in E_s} a(e) \min\{q^{-1}(o(e)), q^{-1}(t(e))\} |(d_\sigma u)(e)|^2 \right)^{1/2} \\ &\quad + \frac{\sqrt{N}}{s} \|u\| \left( \sum_{e \in E_s} a(e) \min\{q^{-1}(o(e)), q^{-1}(t(e))\} |(d_\sigma v)(e)|^2 \right)^{1/2}. \end{aligned} \quad (3.22)$$

**Proof** Using (1.4), (1.6), and the property  $\sigma(\hat{e}) = \overline{\sigma(e)}$ , and recalling that  $W$  is real-valued, we can rewrite (3.20) as

$$J_s = \sum_{x \in V} \sum_{e \in \mathcal{O}_x} \left( 1 - \frac{P(x)}{s} \right)^+ a(e) \left( \sigma(e) u(x) \overline{v(t(e))} - \sigma(\hat{e}) u(t(e)) \overline{v(x)} \right). \quad (3.23)$$

An edge  $e = [x, y] \in E_0$  occurs twice in (3.23): once as  $[x, y]$  and once as  $[y, x]$ . Since  $a([x, y]) = a([y, x])$ , it follows that the contribution of  $e = [x, y]$  and  $\hat{e} = [y, x]$  together in (3.23) is

$$\begin{aligned} &\left( \left( 1 - \frac{P(x)}{s} \right)^+ - \left( 1 - \frac{P(t(e))}{s} \right)^+ \right) a(e) \left( \sigma(e) u(x) \overline{v(t(e))} \right. \\ &\quad \left. - \sigma(\hat{e}) u(t(e)) \overline{v(x)} \right). \end{aligned} \quad (3.24)$$

Using (3.24) and the definition of  $d_\sigma$ , we can rewrite (3.23) as

$$\begin{aligned} J_s &= \sum_{e \in E_s} \left( \left( 1 - \frac{P(o(e))}{s} \right)^+ - \left( 1 - \frac{P(t(e))}{s} \right)^+ \right) a(e) \left( \overline{(d_\sigma v)(e)} u(o(e)) \right. \\ &\quad \left. - (d_\sigma u)(e) \overline{v(o(e))} \right). \end{aligned} \quad (3.25)$$

Using triangle inequality and property

$$|f^+(x) - g^+(x)| \leq |f(x) - g(x)|,$$

from (3.25) we obtain

$$\begin{aligned} |J_s| &\leq \frac{1}{s} \sum_{e \in E_s} a(e) |P(t(e)) - P(o(e))| (|(d_\sigma v)(e)| |u(o(e))| \\ &\quad + |(d_\sigma u)(e)| |v(o(e))|). \end{aligned} \quad (3.26)$$

By (3.19) and (1.8) we get

$$\begin{aligned} |P(t(e)) - P(o(e))| &\leq d_{w,a;q}(t(e), o(e)) \\ &\leq \frac{w^{1/2}(o(e)) \min\{q^{-1/2}(o(e)), q^{-1/2}(t(e))\}}{\sqrt{a(e)}}. \end{aligned} \quad (3.27)$$

Combining (3.26) and (3.27), and using Cauchy–Schwarz inequality together with assumption (1.1), we obtain (3.22).  $\square$

#### Continuation of the proof of Theorem 1.4

Let  $u \in \text{Dom}(H_{\max})$  and  $v \in \text{Dom}(H_{\max})$ . By Lemma 2.6 it follows that  $Hu \in \ell_w^2(V)$  and  $Hv \in \ell_w^2(V)$ . Letting  $s \rightarrow +\infty$  in (3.22) and using (3.2), it follows that  $J_s \rightarrow 0$  as  $s \rightarrow +\infty$ . This, together with (3.21), shows (3.1).  $\square$

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